Sign-rank of k-Hamming Distance is constant

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Joint work with



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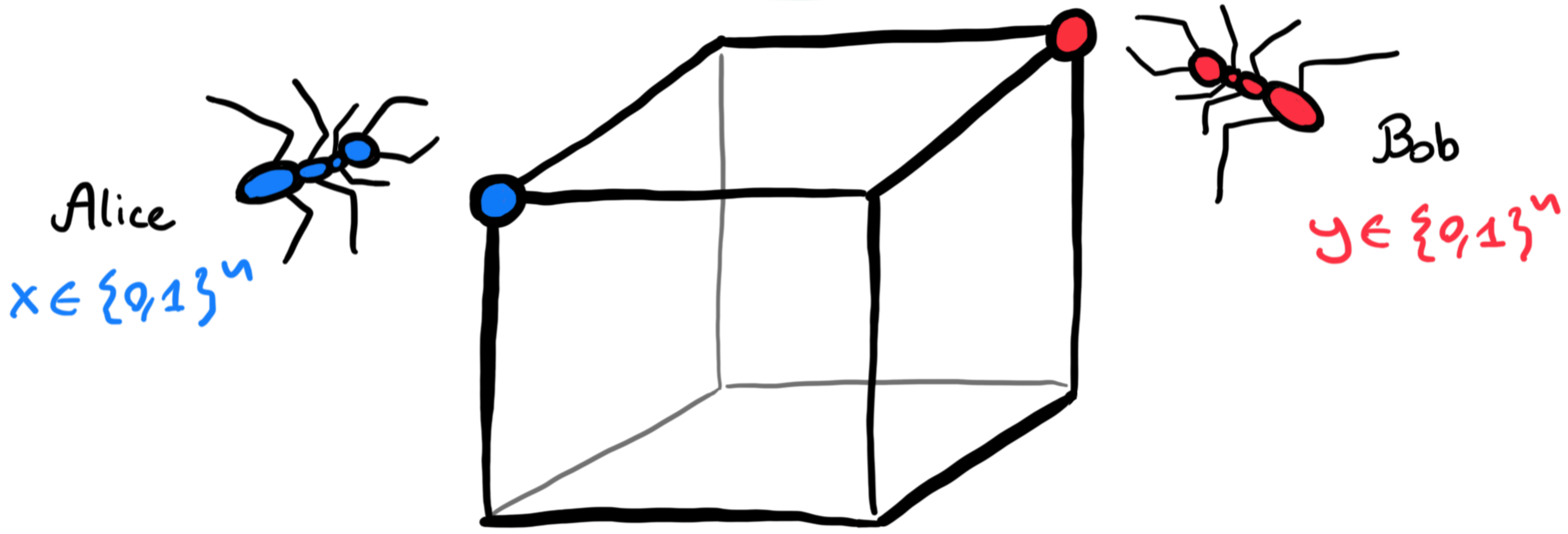
A Communication Problem

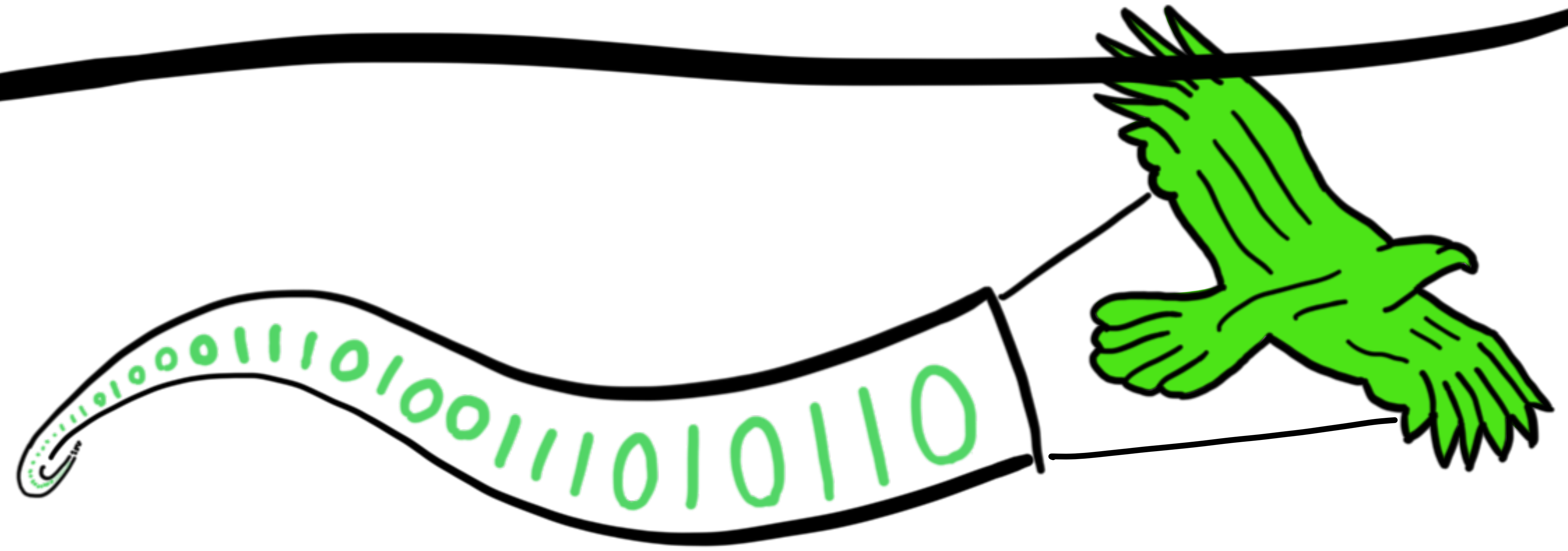
$HD(x, y) \leq k$?

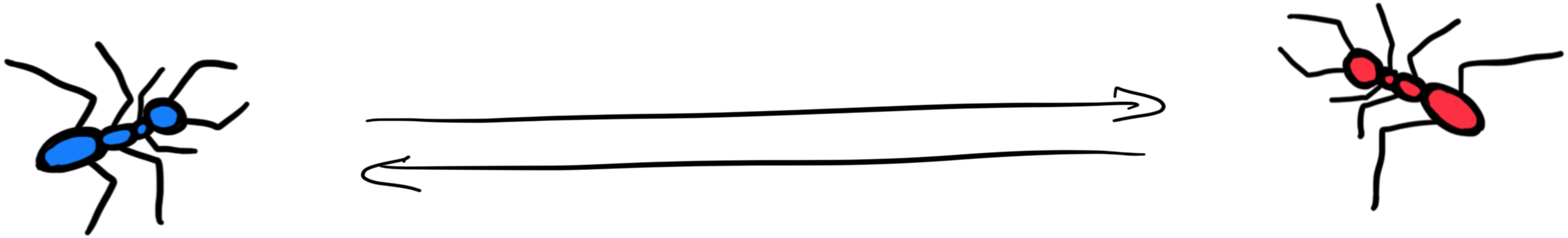


Fact: Deterministically, need $\Omega(n)$ bits!

Yao 76

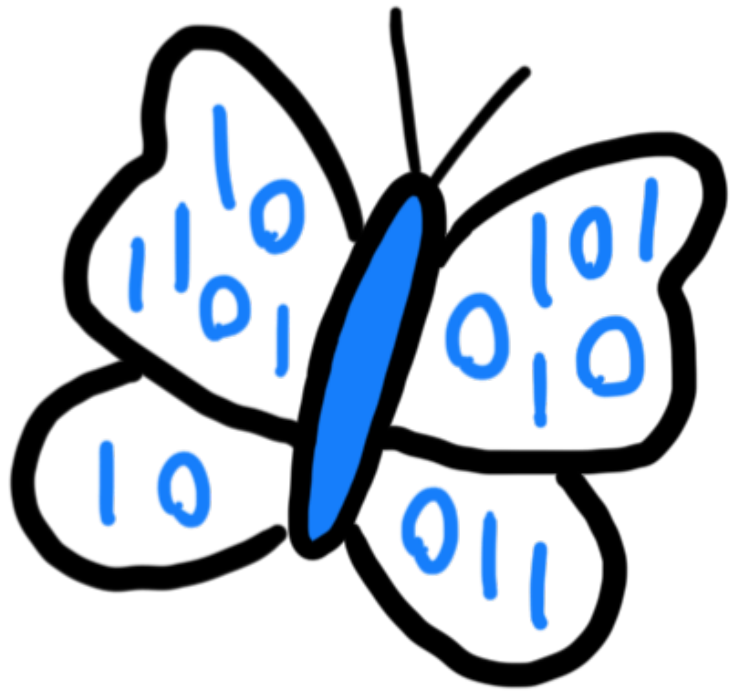
Public Randomness





Constant communication & $\mathbb{P}(\text{correctness}) \geqslant 2/3$ HS2206

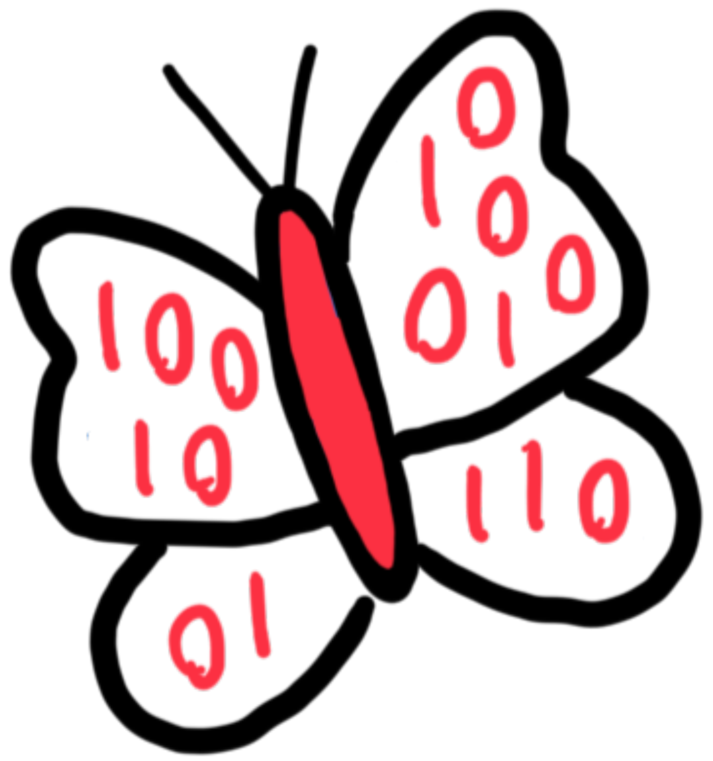
Private Randomness

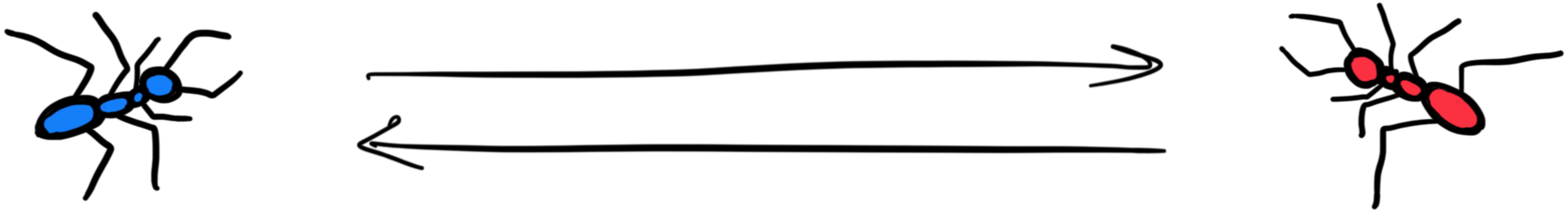


Conjectured: No

HHP+22, HHM23, HZ24, etc.

We prove: YES





Constant communication & $\mathbb{P}(\text{correctness}) > 1/2$?

Sign-Rank

Sign-rank = least rank

$\mathrm{HD}(x, y) \leq 1$?

Yes $\rightarrow \mathbb{R}_{>0}$

No $\rightarrow \mathbb{R}_{<0}$

	000	001	010	011	100	101	110	111
000	+	+	+	−	+	−	−	−
001	+	+	−	+	−	+	−	−
010	+	−	+	+	−	−	+	−
011	−	+	+	+	−	−	−	+
100	+	−	−	−	+	+	+	−
101	−	+	−	−	+	+	−	+
110	−	−	+	−	+	−	+	+
111	−	−	−	+	−	+	+	+

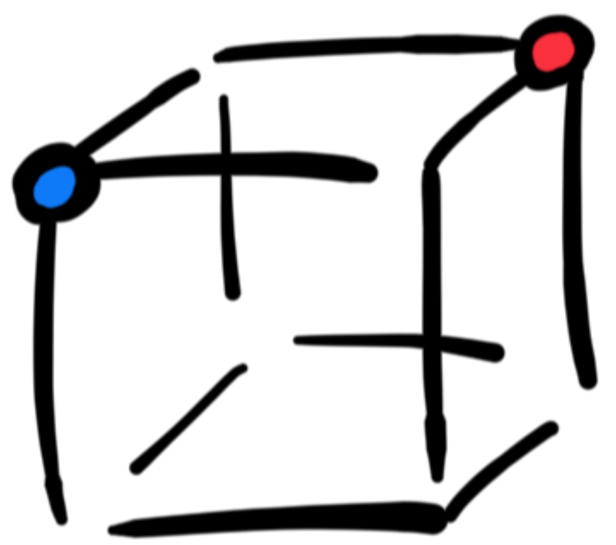
$\in \mathbb{R}^{2^n \times 2^n}$

$$\boxed{\text{communication complexity} = \log(\text{sign-rank}) + O(1)}$$

PS86

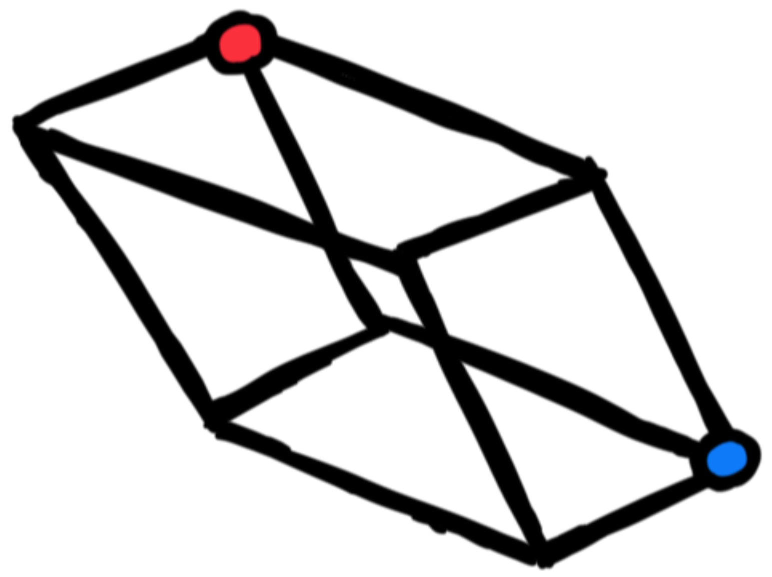
The Plans

Encoding



Dimension n

Projection



Constant Dimension

Polynomial

$$P(\bar{x},\bar{y}) = 0\ ?$$

Constant Degree/Variables

Support-Rank

$$\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$$

Constant Rank

Sign-Rank

$$\begin{pmatrix} + & - \\ - & + \end{pmatrix}$$

Constant Rank

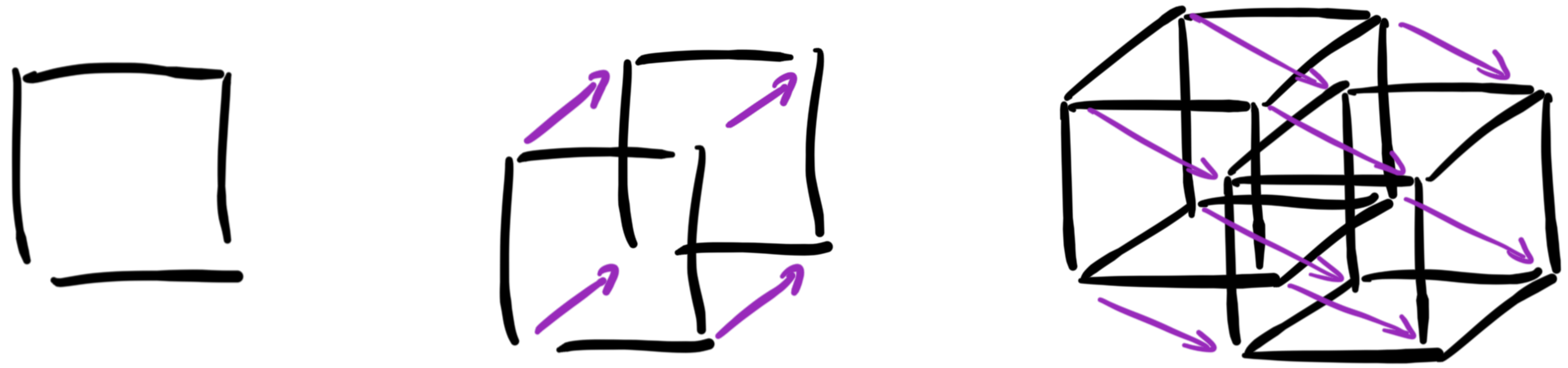
The Case $k=1$

Fact: $\exists\, h : \{0,1\}^n \to \mathbb{R}^2$ such that

$$\|h(x)-h(y)\|_2 = 1 \iff HD(x,y) = 1$$

EHT65

Erdős!



$n=2$ $n=3$ $n=4$

$$\|h(x)-h(y)\|_2 = 1 \iff (x_1-y_1)^2+(x_2-y_2)^2-1 = 0$$

Veronese Map

$$M_{xy} = (x_1 - y_1)^2 + (x_2 - y_2)^2 - 1 = \begin{pmatrix} 1 \\ -x_1 \\ -x_2 \\ x_1^2 + x_2^2 - 1 \end{pmatrix} \cdot \begin{pmatrix} y_1^2 + y_2^2 \\ 2y_1 \\ 2y_2 \\ 1 \end{pmatrix}$$

$$\text{support-rank} \leq \text{rank}(M) = O(1)$$

$$\boxed{\underline{\textbf{Fact:}}\ \text{sign-rank} \leq (\text{support-rank})^2 + 1}$$

Matrix Encoding

$x = 001011$

$$X = \begin{pmatrix} 0 & & & & & 0 \\ & 0 & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & 1 & \\ 0 & & & & & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & & & & & 0 \\ & 1 & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & 0 & \\ 0 & & & & & 1 \end{pmatrix} = Y$$

$y = 010001$

$$\mathrm{rank}(X - Y) = HD(x, y)$$

Reduce dimension & preserve rank?

Random Projection

Lemma: $\exists$ linear $\Pi : \mathbb{R}^{n \times n} \to \mathbb{R}^{k \times k}$

$\forall A \in \{0, \pm 1\}^{n \times n}$ $\operatorname{rank}(\Pi A) = \min(\operatorname{rank}(A), k)$

Conclusion

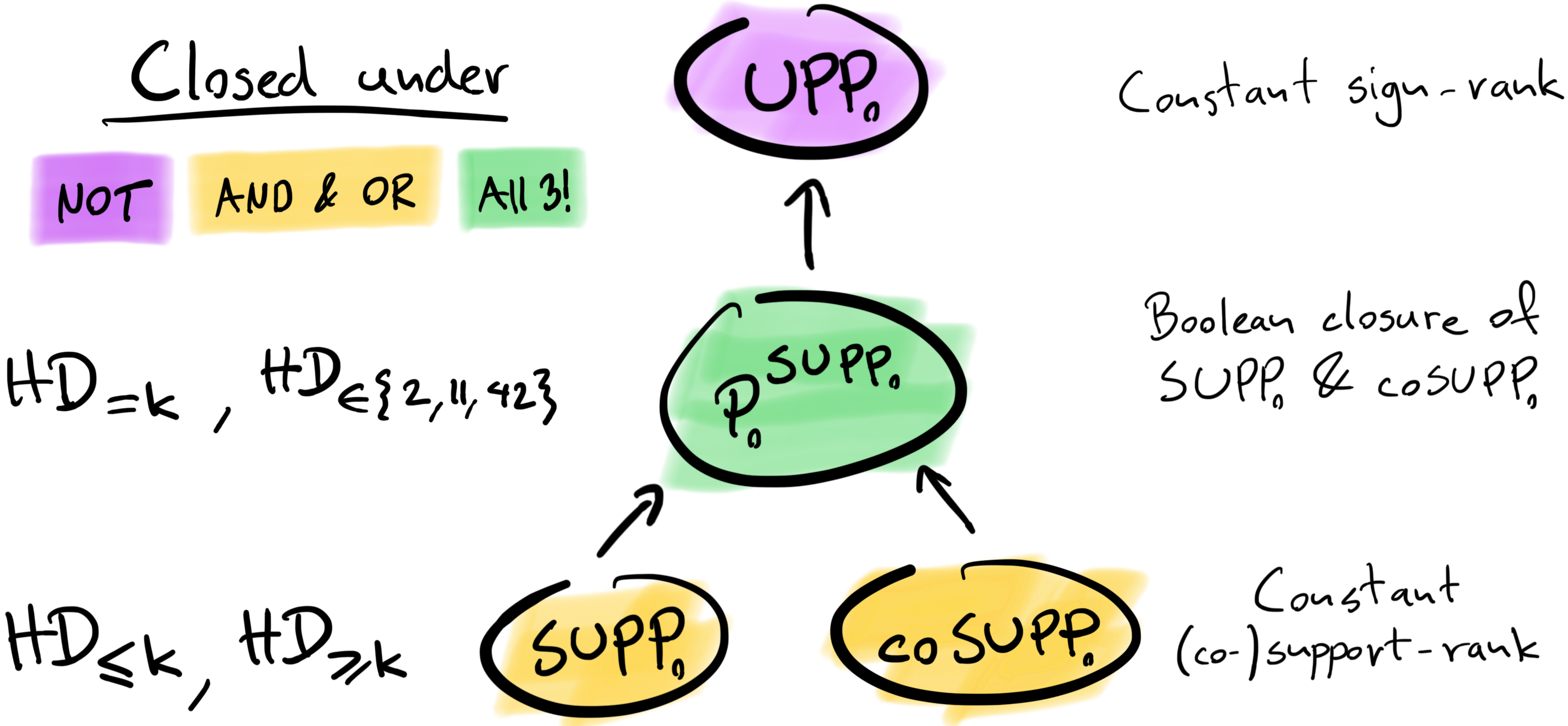
$$HD(x,y) < k \overset{\text{Encoding}}{\Longleftrightarrow} \operatorname{rank}(X - Y) < k$$

$$\Updownarrow \text{ Projection}$$

$$\det(\Pi X - \Pi Y) = 0 \overset{\text{Determinant}}{\Longleftrightarrow} \operatorname{rank}(\Pi X - \Pi Y) < k$$

Main Result $\quad \text{sign-rank}(HD_{<k}) = 2^{O(k)}$

What's SUPP?



Closed under
NOT
AND & OR
All 3!
UPP_0
Constant sign-rank
$HD_{=k}$, $HD_{\in\{2,11,42\}}$
$P_0^{SUPP_0}$
Boolean closure of $SUPP_0$ & $coSUPP_0$
$HD_{\leq k}$, $HD_{\geq k}$
$SUPP_0$
$coSUPP_0$
Constant (co-)support-rank

§ Open Questions

1. Find another constant-cost problem that separates private from public randomness.

2. How exhaustive is our upper-bound method?

3. Relation to other classes like I'_2, $P_0^{RP_0}$, etc ?